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TURBULENT TEMPERATURE FLUCTUATIONS AND TWO-DIMENSIONAL HEAT TRANSFER IN A UNIFORM SHEAR FLOW

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Abstract—Correlation equations for statistically homogeneous fluctuations of velocity and temperature at two points in an infinite uniform shear flow are derived with allowance for a temperature gradient in an arbitrary direction in a plane normal to the flow direction. The initially excited isotropic turbulence decays and becomes anisotropic with time. After Fourier transformations are introduced, the resulting spectral equations are solved for the case of weak turbulence wherein triple correlations are neglected compared with double correlations. Spectra of turbulent heat transfer and temperature fluctuation are calculated. For large nondimensional velocity gradients, the thermal eddy diffusivity in the direction normal to the velocity gradient is larger than that in the direction of the velocity gradient. The thermal eddy diffusivity in the velocity-gradient direction tends to equal the momentum eddy diffusivity at large velocity gradients.

NOMENCLATURE

a, transverse velocity gradient, dU_1/dx_2;
a^*, dimensionless transverse velocity gradient, $(t - t_0)\,dU_1/dx_2$;
b, transverse temperature gradient, $\partial T/\partial x_2$;
c, transverse temperature gradient, $\partial T/\partial x_3$;
J_0, constant that depends on initial conditions;
P, P', arbitrary points;
Pr, Prandtl number, ν/α;
p, instantaneous pressure;
r, distance from P to P';
$\mathbf{r}$, distance vector from P to P';
r_k, component of $\mathbf{r}$;
T, average temperature;
$\tilde{T}$, instantaneous temperature,
T_t, transfer term for temperature fluctuations obtained by integrating $\kappa_1\partial\delta/\partial\kappa_2$ in equation (25) over the angular co ordinates of a wave number sphere;
t, time;
t_0, initial value of t;
U_k, an average velocity component;
$\tilde{u}_k$, instantaneous velocity component;
u_k, fluctuating part of velocity component defined by equation (4);
x_k, space coordinate;
α, thermal diffusivity;
Γ_2, Γ_3, spectrum functions of $\overline{\tau u_2}$ or $\overline{\tau u_3}$ defined by equation (29);
γ_j, Fourier transform of $\overline{\tau u_j}$ defined by equation (15);
γ_i', Fourier transform of $\overline{u_i\tau'}$ defined by equation (16);
Δ, spectrum function of $\overline{\tau^2}$ defined by equation (30);
δ, Fourier transform of $\overline{\tau\tau'}$ defined by equation (17);
δ_{ij}, equals 1 for $i = j$ and equals 0 for $i \neq j$;
ϵ, eddy diffusivity for momentum transfer defined by equation (34);
ϵ_h, eddy diffusivity for heat transfer defined by equation (34);
ζ, Fourier transform of $\overline{\tau p'}$ defined by equation (19);
ζ', Fourier transform of $\overline{p\tau'}$ defined by equation (20);
θ, spherical coordinate in wave-number space;

κ,	wave number;
κ^*,	dimensionless wave number, $\nu^{1/2}(t - t_0)^{1/2} \kappa$;
$\boldsymbol{\kappa}$,	wave number vector;
κ_i,	component of wave number vector;
ν,	kinematic viscosity;
ξ,	dummy variable;
ρ,	density;
τ,	fluctuating part of temperature defined by equation (3);
φ,	spherical coordinate in wave number space;
φ_{ij},	Fourier transform of $\overline{u_i u_j'}$ defined by equation (18).

Subscripts

i, j, k,	values equaling 1, 2, or 3 and designating coordinate directions;
(2), (3),	scalar quantities that arise from the effects of $\partial T/\partial x_2$ or $\partial T/\partial x_3$.

Superscripts

$'$,	point P';
—,	average value;
*,	dimensionless quantity.

INTRODUCTION

PHENOMENOLOGICAL theories of turbulence, which are reviewed in [1], have recently received support from statistical turbulence theory. In a uniform shear flow with decaying turbulence, Deissler [2] found a tendency of the ratio of eddy diffusivities for heat and momentum to approach unity for conditions that correspond roughly to steady channel flow. Developments of this nature do not form a basis for supplanting phenomenological theories, which are the only practical means of organizing quantities of experimental evidence. Rather, statistical theories further the understanding of turbulence and may, in some instances, point the way for new extensions of the phenomenological theories when no experimental evidence is available.

A uniform shear flow is described by a constant gradient of mean velocity in a direction normal to the flow direction. No boundaries are present. Transient turbulence, which is spatially homogeneous, is initially established, for instance, by a wire screen, and the turbulence is later studied when it is weak enough for the triple correlations of velocity or temperature fluctuations to be neglected.

Early statistical investigations of turbulent heat transfer were concerned with the isotropic turbulence that arises in the absence of a mean velocity gradient [3, 4]. For shear flows, numerical values of the velocity correlations were first presented by Deissler [5]. Additional studies of heat transfer, pressure fluctuations and velocity correlations were accomplished by him [2, 6] and Fox [7]. In the present effort, these studies are extended to include the effects of a temperature gradient with components not only in the direction of the velocity gradient (the subject of [2]) but also in the direction normal to both the velocity vector and the velocity gradient. A similar arrangement of vectors occurs in a tube flow with circumferential variations in heat transfer. In the following development, temperature gradient effects are shown to be separable into components; consequently, the results of the present investigation supplement those of [2].

Several features of stronger turbulence are present in weak turbulent shear flows, as shown in [5]. Transfer between eddies of different sizes is present, as is production of turbulence by the action of the mean velocity gradient. The decay of velocity and temperature fluctuations proceeds, however, despite the production effects since they are not strong enough to offset dissipation effects.

ANALYTICAL FORMULATION

The thermal energy equations at two points P and P' can be written, for constant properties, as

$$\frac{\partial \tilde{T}}{\partial t} + \frac{\partial(\tilde{u}_k \tilde{T})}{\partial x_k} = \alpha \frac{\partial^2 \tilde{T}}{\partial x_k \partial x_k} \tag{1}$$

and

$$\frac{\partial \tilde{T}'}{\partial t} + \frac{\partial(\tilde{u}_k' \tilde{T}')}{\partial x_k'} = \alpha \frac{\partial^2 \tilde{T}'}{\partial x_k' \partial x_k'}, \tag{2}$$

where $\tilde{u}_k$ and $\tilde{T}$ are the instantaneous velocity components and temperature. Cartesian components of the position vector $\mathbf{x}$ are designated by the subscript k, which takes on the values 1, 2, and 3. A repeated subscript on a term implies a summation of three terms corresponding to the three values of the subscript. Symbols t and α represent time and thermal diffusivity. A division of instantaneous quantities into steady and fluctuating components is accomplished by setting

$$\tilde{T} = T + \tau \quad (3)$$

and

$$\tilde{u}_k = U_k + u_k. \quad (4)$$

These relations are substituted into equation (1), and the resulting equation is averaged over a large number of systems that are macroscopically the same but have random fluctuating quantities that are spatially homogeneous in a statistical sense (ensemble average). The averaged equations are subtracted from the unaveraged ones with the result

$$\frac{\partial \tau}{\partial t} + U_k \frac{\partial \tau}{\partial x_k} + u_k \frac{\partial T}{\partial x_k} + \frac{\partial(\tau u_k)}{\partial x_k} - \frac{\partial \overline{\tau u_k}}{\partial x_k} = \alpha \frac{\partial^2 \tau}{\partial x_k \partial x_k}, \quad (5)$$

where the overbar indicates an averaged quantity. The average of a fluctuating component is necessarily zero. At point P', the equation corresponding to equation (5) can be visualized from equations (1) and (2). In a similar manner, the Navier–Stokes equations were treated in [5] to yield (at point P′)

$$\frac{\partial u_j'}{\partial t} + u_k' \frac{\partial U_j'}{\partial x_k'} + U_k' \frac{\partial u_j'}{\partial x_k'} + \frac{\partial}{\partial x_k'}(u_j' u_k') - \frac{\partial}{\partial x_k'}\overline{(u_j' u_k')} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j'} + \nu \frac{\partial^2 u_j'}{\partial x_k' \partial x_k'}. \quad (6)$$

An equation for $\overline{\tau u_j'}$ is obtained by multiplying equation (5) by u_j' and equation (6) by τ, adding, and averaging the resulting equation. In the interest of brevity, the turbulence is assumed weak at this point in the analysis so that the triple correlations that arise are neglected compared with the double correlations. None of the omitted triple correlations is different from that in [2]. The averaged equation is

$$\frac{\partial \overline{\tau u_j'}}{\partial t} + U_k \frac{\partial \overline{\tau u_j'}}{\partial x_k} + \overline{u_k u_j'}\frac{\partial T}{\partial x_k} + \overline{\tau u_k'}\frac{\partial U_j'}{\partial x_k'} + U_k' \frac{\partial \overline{\tau u_j'}}{\partial x_k'} = -\frac{1}{\rho}\frac{\partial \overline{\tau p'}}{\partial x_j'} + \nu \frac{\partial^2 \overline{\tau u_j'}}{\partial x_k' \partial x_k'} + \alpha \frac{\partial^2 \overline{\tau u_j'}}{\partial x_k \partial x_k}, \quad (7)$$

where the independence of fluctuating quantities at one point from the position of the other point has been utilized in placing the quantities inside the spacial derivative signs. In homogeneous turbulence, $(\partial/\partial x_k')_{x_k} = \partial/\partial r_k$ and $(\partial/\partial x_k)_{x_k'} = -\partial/\partial r_k$ if $\mathbf{x}' = \mathbf{x} + \mathbf{r}$. In the present case of a single steady velocity component U_1 and one velocity gradient dU_1/dx_2, a simplifying relation exists:

$$(U_k' - U_k)\frac{\partial}{\partial r_k}\overline{\tau u_j'} = \frac{dU_1}{dx_2} r_2 \frac{\partial}{\partial r_1}\overline{\tau u_j'}, \quad (8)$$

which reduces equation (7) to

$$\frac{\partial}{\partial t}\overline{\tau u_j'} + \frac{dU_1}{dx_2} r_2 \frac{\partial}{\partial r_1}\overline{\tau u_j'} + \overline{u_k u_j'}\frac{\partial T}{\partial x_k} + \overline{\tau u_2'}\,\delta_{1j}\frac{dU_1}{dx_2} = -\frac{1}{\rho}\frac{\partial}{\partial r_j}\overline{\tau p'} + (\alpha + \nu)\frac{\partial^2 \overline{\tau u_j'}}{\partial r_k \partial r_k}, \quad (9)$$

where $\delta_{1j} = 1$ for $j = 1$ and 0 for $j \neq 1$.

A similar procedure applied to the equations corresponding to equations (5) and (6) at P' and P yields

$$\frac{\partial}{\partial t}\overline{u_i \tau'} + \overline{u_2 \tau'}\,\delta_{i1}\frac{dU_1}{dx_2} + \overline{u_i u_k'}\frac{\partial T}{\partial x_k} + \frac{dU_1}{dx_2} r_2 \frac{\partial}{\partial r_1}\overline{u_i \tau'} = \frac{1}{\rho}\frac{\partial}{\partial r_i}\overline{p\tau'} + (\alpha + \nu)\frac{\partial^2 \overline{u_i \tau'}}{\partial r_k \partial r_k}, \quad (10)$$

and, likewise, equation (5) at P and the corresponding equation at P' yield

$$\frac{\partial}{\partial t}\overline{\tau\tau'} + \frac{dU_1}{dx_2} r_2 \frac{\partial}{\partial r_1}\overline{\tau\tau'} + \frac{\partial T}{\partial x_k}(\overline{u_k \tau'} + \overline{\tau u_k'}) = 2\alpha \frac{\partial^2 \overline{\tau\tau'}}{\partial r_k \partial r_k}. \quad (11)$$

An additional equation is obtained by applying $\partial/\partial x_j'$ to equation (6) and noting the continuity equation $\partial u_j'/\partial x_j' = 0$. This produces

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_j' \partial x_j'} = -2\frac{\partial u_k'}{\partial x_j'}\frac{\partial U_j'}{\partial x_k'} - \frac{\partial^2(u_j' u_k')}{\partial x_j' \partial x_k'} + \frac{\partial^2 \overline{u_j' u_k'}}{\partial x_j' \partial x_k'}. \tag{12}$$

Multiplying equation (12) by τ, averaging, and introducing $r_j = x_j' - x_j$ give

$$\frac{1}{\rho}\frac{\partial^2 \overline{\tau p'}}{\partial r_j \partial r_j} = -2\frac{\mathrm{d}U_1}{\mathrm{d}x_2}\frac{\partial \overline{\tau u_2'}}{\partial r_1}. \tag{13}$$

Similarly,

$$\frac{1}{\rho}\frac{\partial^2 \overline{p\tau'}}{\partial r_i \partial r_i} = 2\frac{\mathrm{d}U_1}{\mathrm{d}x_2}\frac{\partial \overline{u_2 \tau'}}{\partial r_1}. \tag{14}$$

Fourier transforms are introduced:

$$\overline{\tau u_j'}(\mathbf{r}) = \int_{-\infty}^{\infty} \gamma_j(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{15}$$

$$\overline{u_i \tau'}(\mathbf{r}) = \int_{-\infty}^{\infty} \gamma_i'(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{16}$$

$$\overline{\tau\tau'}(\mathbf{r}) = \int_{-\infty}^{\infty} \delta(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{17}$$

$$\overline{u_i u_j'}(\mathbf{r}) = \int_{-\infty}^{\infty} \varphi_{ij}(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{18}$$

$$\overline{\tau p'}(\mathbf{r}) = \int_{-\infty}^{\infty} \zeta(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{19}$$

$$\overline{p\tau'}(\mathbf{r}) = \int_{-\infty}^{\infty} \zeta'(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{r}]\, \mathrm{d}\boldsymbol{\kappa}, \tag{20}$$

where $|\boldsymbol{\kappa}| = \kappa$ is the wave number, which can be interpreted as the reciprocal of an eddy size.

The Fourier transforms of equations (9) and (13) are

$$\frac{\partial \gamma_j}{\partial t} - \frac{\mathrm{d}U_1}{\mathrm{d}x_2}\kappa_1\frac{\partial \gamma_j}{\partial \kappa_2} + \varphi_{kj}\frac{\partial T}{\partial x_k} + \delta_{1j}\gamma_2\frac{\mathrm{d}U_1}{\mathrm{d}x_2} = -\frac{1}{\rho}i\kappa_j\zeta - (\alpha+\nu)\kappa^2\gamma_j \tag{21}$$

and

$$-\frac{1}{\rho}i\kappa_j\zeta = 2\frac{\kappa_1\kappa_j}{\kappa^2}\gamma_2\frac{\mathrm{d}U_1}{\mathrm{d}x_2}, \tag{22}$$

where $\mathrm{d}U_1/\mathrm{d}x_2$ and $\partial T/\partial x_k$ are constants. Equation (22) can be subtracted from equation (21) and ζ can thus be eliminated from the equations. A similar procedure applied to equations (10) and (14) yields an equation of the same form for γ_i'. In the present study, the case of $r = 0$, $\partial T/\partial x_2 \neq 0$, and $\partial T/\partial x_3 \neq 0$ is considered so that $\gamma_i' = \gamma_j$. Symbols $a = \mathrm{d}U_1/\mathrm{d}x_2$, $b = \partial T/\partial x_2$, and $c = \partial T/\partial x_3$ are introduced so that the final equations become

$$\frac{\partial \gamma_2}{\partial t} - a\kappa_1\frac{\partial \gamma_2}{\partial \kappa_2} = -b\varphi_{22} - c\varphi_{23} + \left[2a\frac{\kappa_1\kappa_2}{\kappa^2} - \left(\frac{1}{Pr}+1\right)\nu\kappa^2\right]\gamma_2, \tag{23}$$

$$\frac{\partial \gamma_3}{\partial t} - a\kappa_1\frac{\partial \gamma_3}{\partial \kappa_2} = -b\varphi_{32} - c\varphi_{33} + 2a\frac{\kappa_1\kappa_3}{\kappa^2}\gamma_2 - \left(\frac{1}{Pr}+1\right)\nu\kappa^2\gamma_3, \tag{24}$$

and

$$\frac{\partial \delta}{\partial t} - a\kappa_1\frac{\partial \delta}{\partial \kappa_2} = -2b\gamma_2 - 2c\gamma_3 - 2\alpha\kappa^2\delta, \tag{25}$$

where equation (25) is the Fourier transform of equation (11).

SOLUTION OF SPECTRAL EQUATIONS

Isotropic turbulence and zero temperature fluctuations are assumed as initial conditions. Expressions for φ_{22} and φ_{33} that satisfy these initial conditions have been reported in [5] and [7]. The latter is

$$\left.\begin{aligned}\varphi_{33} = {} & \frac{J_0\{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2\}^2}{12\pi^2(\kappa_1^2+\kappa_3^2)} \\ & \times \exp\{-2\nu(t-t_0)[\kappa^2 + \tfrac{1}{3}a^2\kappa_1^2(t-t_0)^2 + a\kappa_1\kappa_2(t-t_0)]\} \\ & \times \left\{\frac{\kappa_1^2}{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2} + \frac{\kappa_3^2\kappa_2^2}{\kappa^4} + \frac{2\kappa_2\kappa_3^2}{(\kappa_1^2+\kappa_3^2)^{1/2}\kappa^2}\left[\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right]\right. \\ & \left. + \frac{\kappa_3^2}{(\kappa_1^2+\kappa_3^2)}\left[\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right]^2\right\},\end{aligned}\right\} \tag{26}$$

where J_0 and t_0 are constants that depend on the initial conditions.

In [7], the solution for $\varphi_{23} = \varphi_{32}$, although nonzero, was found to produce a zero value of $\overline{u_2 u_3}$, which was consistent with the lack of a velocity gradient in the x_2,x_3-plane. Equations (23) and (24) of this investigation have been solved for the effects of φ_{23} by omission of the terms containing φ_{22} and φ_{33}. (The linearity of the equations permits the addition of solutions.) Zero contributions to $\overline{\tau u_2}$ and $\overline{\tau u_3}$ are obtained from the direct effects of φ_{23}; however, an indirect effect that does contribute to $\overline{\tau u_3}$ enters equation (24) in the fifth term. This contribution can be traced to the expression of the pressure effect ζ in terms of γ_2 in equation (22). The remaining portion of γ_2 that contributes to $\overline{\tau u_2}$ is the same as that reported in [2]; it is not repeated herein. In the following solutions to equations (24) and (25), only those expressions that contribute to $\overline{\tau u_3}$ or $\overline{\tau^2}$ are shown.

For $Pr = 1$, the Fourier transform of $\overline{\tau u_3}$ is

$$\left.\begin{aligned}
\gamma_3 = {} & \frac{J_0 c\{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2\}^2}{12\pi^2 a\kappa_1(\kappa_1^2+\kappa_3^2)} \\
& \times \exp\{-2\nu(t-t_0)\,[\kappa^2 + a\kappa_1\kappa_2(t-t_0) + \tfrac{1}{3}a^2\kappa_1^2(t-t_0)^2]\} \\
& \times \Bigg\{-\frac{a\kappa_1^3(t-t_0)}{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2} \\
& + \frac{\kappa_3^2\kappa_2^2}{(\kappa_1^2+\kappa_3^2)^{1/2}\kappa^2}\left[\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right] \\
& + \frac{\kappa_3^2\kappa_2}{(\kappa_1^2+\kappa_3^2)}\left[\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right]^2\Bigg\}.
\end{aligned}\right\} \quad (27a)$$

For $Pr \neq 1$, γ_3 takes the form

$$\left.\begin{aligned}
\gamma_3 = {} & -\frac{J_0 c\{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2\}^2}{12\pi^2 a\kappa_1(\kappa_1^2+\kappa_3^2)} \\
& \exp\Bigg\{\frac{[(1/Pr)-1]\nu\kappa_2}{a\kappa_1}\left(\kappa_1^2 + \frac{\kappa_2^2}{3} + \kappa_3^2\right) \\
& - 2\nu(t-t_0)[\kappa^2 + a\kappa_1\kappa_2(t-t_0) + \tfrac{1}{3}a^2\kappa_1^2(t-t_0)^2]\Bigg\} \int_{\kappa_2}^{\kappa_2 + a\kappa_1(t-t_0)} \\
& \exp\left\{\frac{-[(1/Pr)-1]\nu}{a\kappa_1}\,\xi\left(\kappa_1^2 + \frac{\xi^2}{3} + \kappa_3^2\right)\right\} \\
& \times \Bigg\{\frac{\kappa_1^2}{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2} \\
& + \kappa_3^2\Bigg[\frac{\xi}{\kappa_1^2 + \xi^2 + \kappa_3^2} + \frac{1}{(\kappa_1^2+\kappa_3^2)^{1/2}} \\
& \left(\tan^{-1}\frac{\xi}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right)\Bigg] \\
& \times \Bigg[\frac{\kappa_2}{\kappa^2} + \frac{1}{(\kappa_1^2+\kappa_3^2)^{1/2}}\left(\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right)\Bigg]\Bigg\}\, d\xi.
\end{aligned}\right\} \quad (27b)$$

These expressions for γ_2 and γ_3 verify the fact that the turbulent heat-transfer components $\overline{\tau u_2}$ and $\overline{\tau u_3}$ arise from temperature gradient components in the respective directions $\partial T/\partial x_2$ and $\partial T/\partial x_3$.

The transform of the portion of $\overline{\tau^2}$ that arises from $\partial T/\partial x_3$ is, for $Pr = 1$,

$$\left.\begin{aligned}
\delta_{(3)} = {} & \frac{J_0 c^2\{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2\}^2}{12\pi^2 a^2\kappa_1^2(\kappa_1^2+\kappa_3^2)} \\
& \times \exp\{-2\nu(t-t_0)\,[\kappa^2 + a\kappa_1\kappa_2(t-t_0) + \tfrac{1}{3}a^2\kappa_1^2(t-t_0)^2]\} \\
& \times \Bigg\{\frac{a^2\kappa_1^4(t-t_0)^2}{\kappa_1^2 + [\kappa_2 + a\kappa_1(t-t_0)]^2 + \kappa_3^2} \\
& + \frac{\kappa_3^2\kappa_2^2}{\kappa_1^2+\kappa_3^2}\left[\tan^{-1}\frac{\kappa_2}{(\kappa_1^2+\kappa_3^2)^{1/2}} - \tan^{-1}\frac{\kappa_2 + a\kappa_1(t-t_0)}{(\kappa_1^2+\kappa_3^2)^{1/2}}\right]^2\Bigg\}.
\end{aligned}\right\} \quad (28)$$

The other portion $\delta_{(2)}$ that is a result of $\partial T/\partial x_2$ is the same as that reported in [2].

The convenience of interpreting functions of κ instead of functions of $\boldsymbol{\kappa}$ was pointed out by Batchelor [8]. Following his suggestion, the integrations that lead to $\overline{\tau^2_{(3)}}$ and $\overline{\tau u_3}$ are accomplished in two steps, the first of which involves integrating over the angular coordinates of a wave-number sphere:

$$\Gamma_3(\kappa) = \int_0^{\pi}\int_0^{2\pi} \gamma_3(\kappa, \varphi, \theta)\kappa^2 \sin\theta \, d\varphi \, d\theta \quad (29)$$

and

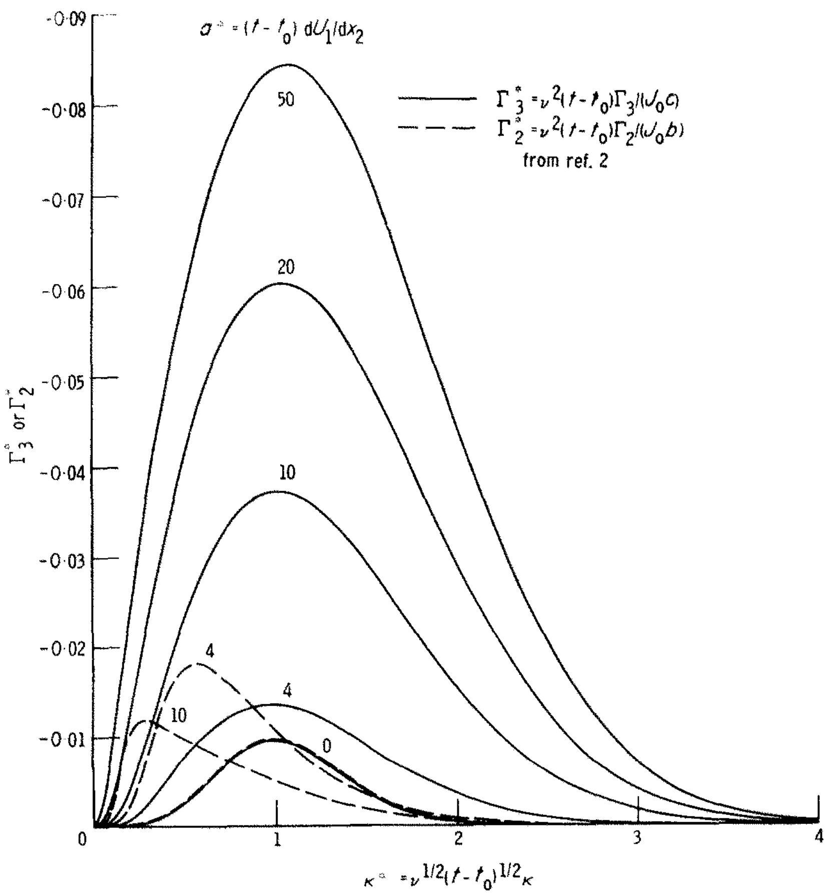
$$\Delta_{(3)}(\kappa) = \int_0^{\pi}\int_0^{2\pi} \delta_{(3)}(\kappa, \varphi, \theta)\kappa^2 \sin\theta \, d\varphi \, d\theta. \quad (30)$$

A display of Γ_3 and $\Delta_{(3)}$ shows how the contributions to $\overline{\tau u_3}$ and $\overline{\tau^2_{(3)}}$ are distributed among wave numbers or eddy sizes, since

$$\overline{\tau u_3} = \int_0^{\infty} \Gamma_3 \, d\kappa \text{ and } \overline{\tau^2_{(3)}} = \int_0^{\infty} \Delta_{(3)} \, d\kappa. \quad (31)$$

COMPUTED SPECTRA

Numerically calculated spectra of $\overline{\tau u_3}$ and $\overline{\tau^2_{(3)}}$ are displayed in dimensionless form in Figs. 1 and 2 for several values of the dimensionless velocity gradient a^*. Since time enters all the dimensionless representations, the curves for various a^* show the effect of velocity gradient on



(a) Prandtl number, 1.

FIG. 1(a). Dimensionless spectra of $\overline{\tau u_3}$ and $\overline{\tau u_2}$ for uniform transverse velocity and temperature gradients and for various Prandtl numbers.

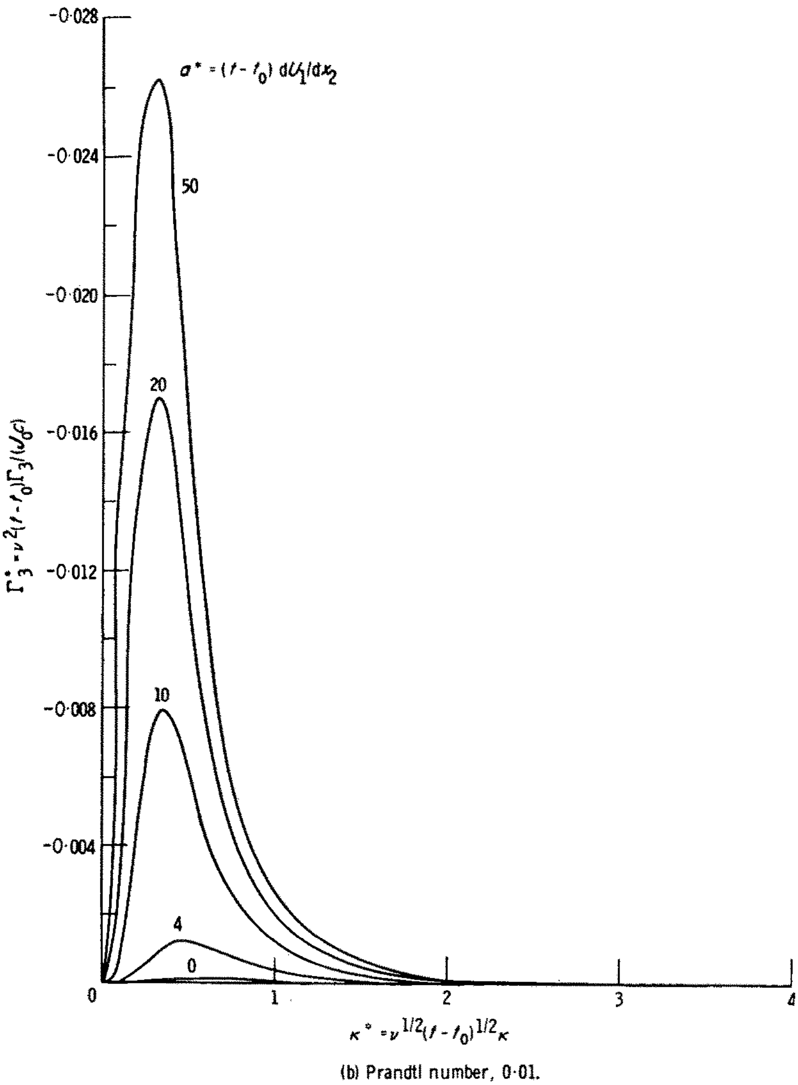


FIG. 1(b). Dimensionless spectra of $\overline{\tau u_3}$ and $\overline{\tau u_2}$ for uniform transverse velocity and temperature gradients and for various Prandtl numbers.

the spectra at any given time while the turbulence decays. Dashed curves correspond to those in [2] because of separability of solutions. For large Prandtl numbers, the spectra of $\overline{\tau u_3}$ in Fig. 1 peak at large wave numbers (small eddy sizes).

Isotropic spectra ($a^* = 0$) in Fig. 1 are the same for $\overline{\tau u_3}$ and $\overline{\tau u_2}$, as previously reported in [4]. The behavior of the peaks of the spectra of $\overline{\tau u_3}$ and $\overline{\tau u_2}$ is similar to that of the respective production terms $c\varphi_{33}$ and $b\varphi_{22}$ in equations (23) and (24); φ_{22} decreases and shifts toward lower wave numbers as the velocity gradient increases, whereas φ_{33} increases markedly with little shift (see Fig. 5 reference 5, and Fig. 2 reference 7).

A shift to higher wave numbers in the spectra of $\overline{\tau^2}$ with increasing velocity gradient is evident in Fig. 2 both at the peak and at moderate values

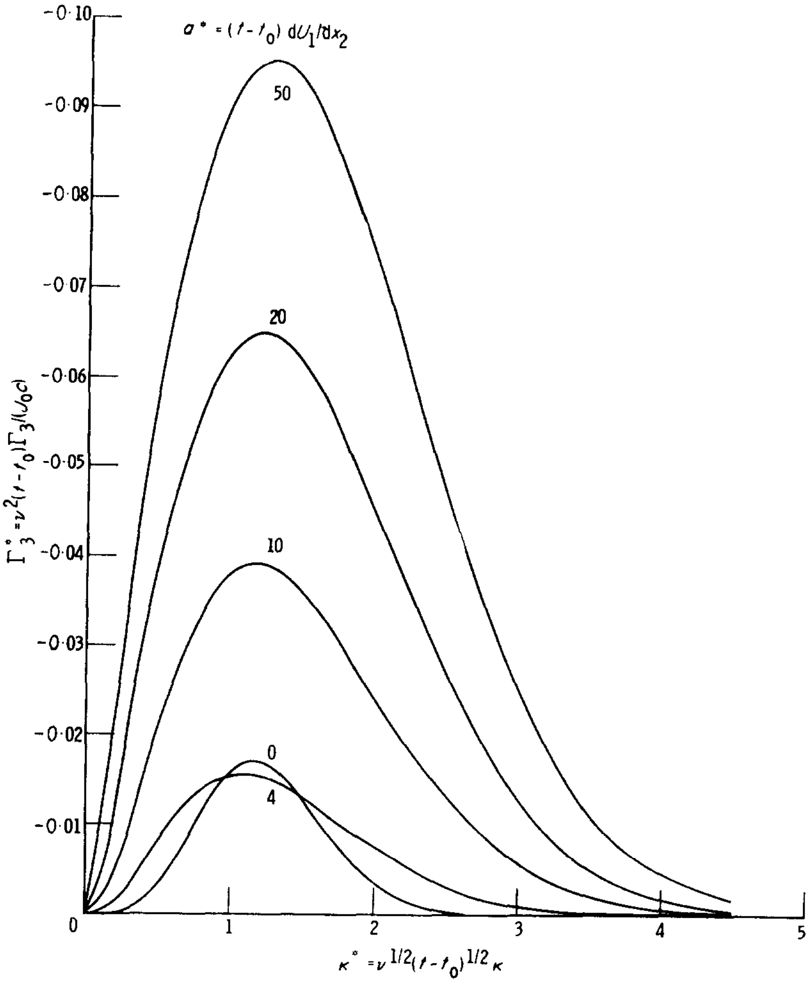


FIG. 1(c). Dimensionless spectra of $\overline{\tau u_3}$ and $\overline{\tau u_2}$ for uniform transverse velocity and temperature gradients and for various Prandtl numbers.

on the high wave number side; the shift results in an elongation of the spectra toward high wave numbers. This spectral change is evidently due to a transfer of activity from low wave numbers (large eddies) to high wave numbers (small eddies) by the action of the second term in equation (25), which is known as the transfer term. The name stems from the Fourier transform relation

$$r_2 \frac{\partial \overline{\tau\tau'}}{\partial r_1} = -\int_{-\infty}^{\infty} \kappa_1 \frac{\partial \delta}{\partial \kappa_2} \exp\,[i\boldsymbol{\kappa}.\mathbf{r}]\,d\boldsymbol{\kappa}, \qquad (32)$$

which becomes, for $r = 0$,

$$\int_{-\infty}^{\infty} \kappa_1 \frac{\partial \delta}{\partial \kappa_2}\, d\boldsymbol{\kappa} = 0. \qquad (33)$$

Similar results can be obtained from corresponding terms in equations (23) and (24). Thus, these terms contribute nothing to $\partial\overline{\tau u_3}/\partial t$, $\partial\overline{\tau u_2}/\partial t$, and $\partial\overline{\tau^2}/\partial t$, but they do alter spectral distributions.

The integration shown in equation (33) can be accomplished in two steps by first integrating

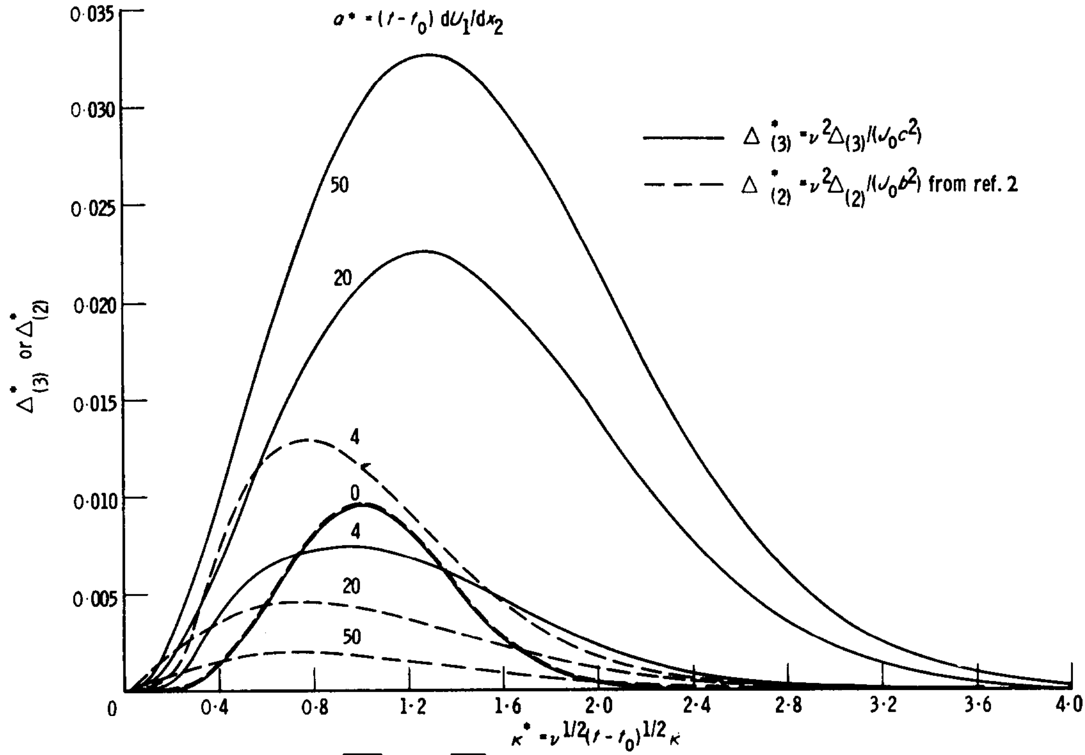


FIG. 2. Dimensionless spectra of $\overline{\tau^2_{(3)}}$ and $\overline{\tau^2_{(2)}}$ for uniform transverse velocity and temperature gradients. Prandtl number, 1.

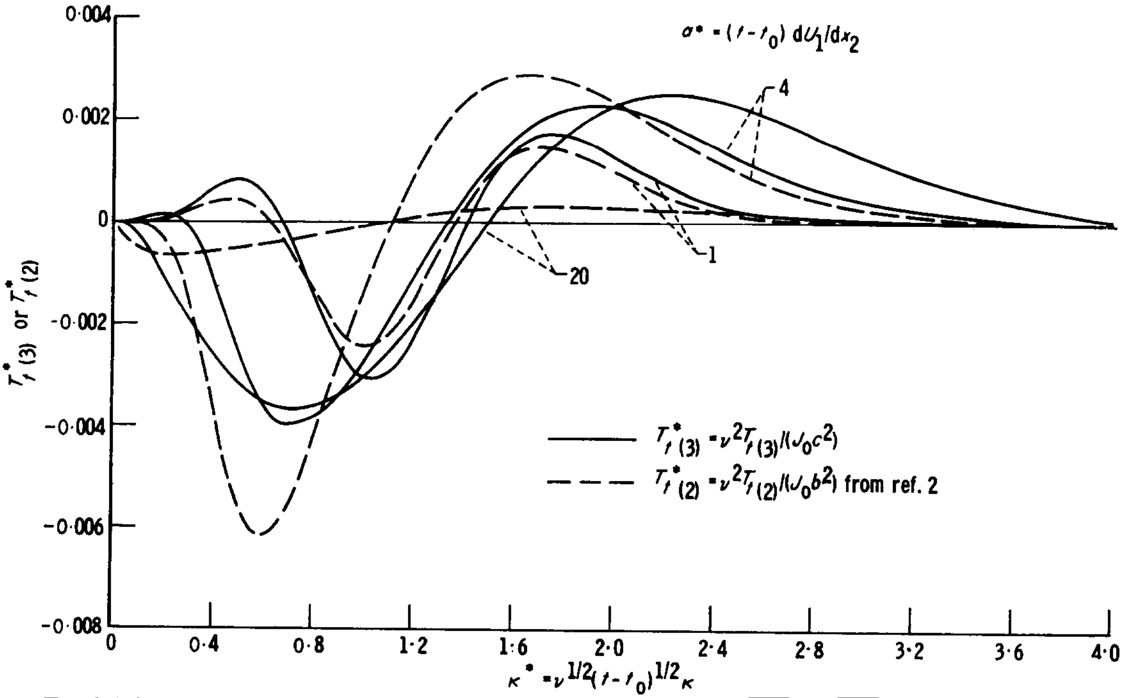


FIG. 3. Dimensionless spectra of transfer terms in spectral equations for $\overline{\tau^2_{(3)}}$ and $\overline{\tau^2_{(2)}}$. Prandtl number, 1.

over the angular coordinates of a wave-number sphere and then integrating over the wave numbers. Of course, the second step yields a trivial result, but the first result is a spectral transfer function. For $Pr = 1$, the integrated transfer term corresponding to $\overline{\tau^2}$ is shown in Fig. 3. Most of the transfer of activity is out of the low-wave-number spectrum and into the high-wave-number spectrum, but some reverse transfer occurs at low wave numbers and low velocity gradients. Deissler [2] attributed this activity transfer to a vortex-stretching process, which might also involve vortex compression at low velocity gradients and thereby produce some reverse transfer.

PRODUCTION, TEMPERATURE FLUCTUATION, AND CONDUCTION SPECTRA

Production of temperature fluctuations by the third and fourth terms in equation (25) is interpreted as a result of the action of the temperature gradient on the respective turbulent heat transfer, $\overline{\tau u_2}$ and $\overline{\tau u_3}$. Conduction or dissipation in the last term reduces local temperature peaks by molecular heat conduction away from hot spots. Production and conduction terms can be integrated over a wave-number sphere to yield spectral distributions in the same manner as that used to obtain the temperature fluctuation spectra of $\overline{\tau^2}$ in Fig. 2. After normalization of the peak values to unity, all three spectra are shown in Fig. 4 for $Pr = 1$ and a high velocity gradient ($a^* = 50$). Actually, two sets of spectra corresponding to the separate effects of $\partial T/\partial x_3$ and $\partial T/\partial x_2$ (from [2]) are displayed in Fig. 4. For low velocity gradients, the three spectra are close together, like those in Fig. 4 of [2]. For a large velocity gradient, production, fluctuation, and conduction spectra in the present Fig. 4 peak at successively greater wave numbers.

All these effects take place as the turbulence and the turbulent temperature fluctuations decay. Fluctuations are produced in the large eddies (low wave numbers), transferred to the small eddies (high wave numbers), and finally dissipated by molecular conduction.

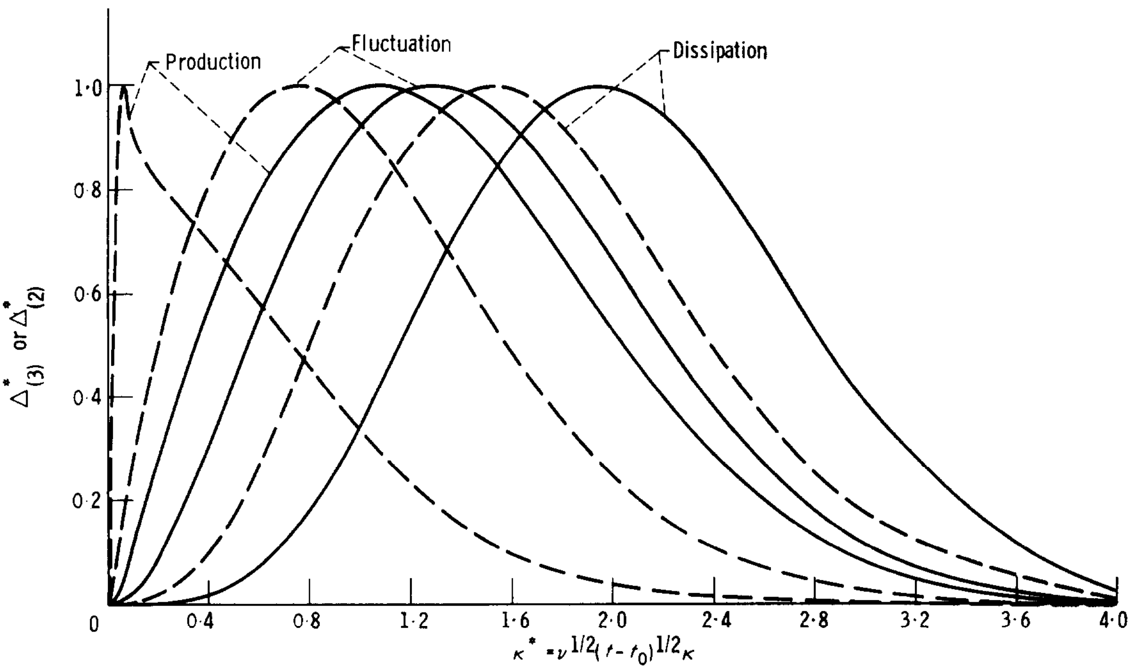


FIG. 4. Comparison of production, temperature fluctuation, and conduction spectra from spectral equations for $\overline{\tau^2}_{(3)}$ and $\overline{\tau^2}_{(2)}$ (solid and dashed curves, respectively) at large velocity gradient. Prandtl number, 1. $a^* = (t - t_0)\,\mathrm{d}U_1/\mathrm{d}x_2 = 50$. (Curves normalized to same height.)

TEMPERATURE–VELOCITY CORRELATION COEFFICIENT

Corrsin [3] introduced a temperature–velocity correlation coefficient that is modified herein to account for the separate effects of $\partial T/\partial x_2$ and $\partial T/\partial x_3$. Two dimensionless coefficients are utilized,

$$\frac{\overline{\tau u_2}}{(\overline{\tau^2_{(2)}}\,\overline{u_2^2})^{1/2}} \text{ and } \frac{\overline{\tau u_3}}{(\overline{\tau^2_{(3)}}\,\overline{u_3^2})^{1/2}},$$

the former being the same as that presented in [2]. The latter coefficient has been calculated from integrals of the curves in Figs. 1 and 2 and those in Fig. 2 [7], all for $Pr = 1$. Fig. 5 is a display of both correlations as a function of velocity gradient, starting with the perfect correlation value of -1 that was obtained in [4] for isotropic turbulence ($a^* = 0$). In the range $0 \leq a^* \leq 50$, one correlation coefficient

$$\frac{\overline{\tau u_3}}{(\overline{\tau^2_{(3)}}\,\overline{u_3^2})^{1/2}}$$

achieves an asymptotic value of $-0{\cdot}9$ whereas the other, by decreasing monotonically, shows a continuous loss of correlation between the temperature and velocity fluctuations as a^* increases.

That the correlation coefficients in Fig. 5 are independent of the temperature gradients is noteworthy. This independence is lacking in the conventional coefficients that utilize

$$\overline{\tau^2}\,(= \overline{\tau^2_{(2)}} + \overline{\tau^2_{(3)}})$$

in place of $\overline{\tau^2_{(2)}}$ or $\overline{\tau^2_{(3)}}$ in the present formulation. In fact, the conventional coefficients are functions of both the velocity gradient a^* and the *unrelated* temperature gradient; for example,

$$\frac{\overline{\tau u_3}}{(\overline{\tau^2}\,\overline{u_3^2})^{1/2}}$$

is a function of a^* and $\partial T/\partial x_2$, as is evident from the solutions of the spectral equations. These misleading functional relationships are absent from the present correlation coefficients, which are functions of a^* alone.

The dimensionless forms of $\overline{\tau^2_{(2)}}$ and $\overline{\tau^2_{(3)}}$ are displayed in Fig. 6.

EDDY DIFFUSIVITIES

The eddy diffusivities of momentum and heat (in the x_2- and x_3-directions) are defined by

$$\epsilon = -\frac{\overline{u_1 u_2}}{\mathrm{d}U_1/\mathrm{d}x_2}, \quad \epsilon_{h(2)} = -\frac{\overline{\tau u_2}}{\partial T/\partial x_2},$$
$$\epsilon_{h(3)} = -\frac{\overline{\tau u_3}}{\partial T/\partial x_3} \tag{34}$$

Ratios of eddy diffusivities play a large part in phenomenological theories of steady turbulent flows. A unity value of $\epsilon_{h(2)}/\epsilon$ produces the best agreement between experiment and analysis for Prandtl numbers that are not too low [1]. In the transient turbulence analysis of [2], Deissler obtained a similar tendency of $\epsilon_{h(2)}/\epsilon$ toward unity at high values of a^* which were found to correspond roughly to steady turbulent flows. Recent phenomenological analysis [9], [10] of circumferential variations of heat transfer in round tubes are based on an assumption of equal eddy diffusivities in the radial and circumferential directions; that is, $\epsilon_{h(2)} = \epsilon_{h(3)}$ in the present notation.

A dimensionless eddy diffusivity

$$\nu^{5/2}(t - t_0)^{3/2}\,\epsilon_{h(3)}/J_0$$

can be obtained by integrating the curves in Fig. 1. Integration of the curves in [5] for ϵ is also necessary for the calculation of $\epsilon_{h(3)}/\epsilon$, which is displayed in Fig. 7 along with $\epsilon_{h(2)}/\epsilon$ from [2]. Although the curves for the two ratios are not widely separated at low velocity gradients, which are near the isotropic case ($a^* = 0$), large velocity gradients produce values of $\epsilon_{h(3)}/\epsilon$ that are two orders of magnitude greater than values of $\epsilon_{h(2)}/\epsilon$, except for low Prandtl numbers.

The relative magnitudes of $\epsilon_{h(3)}$ and $\epsilon_{h(2)}$ can be compared with the magnitudes of the turbulent velocity fluctuations (or turbulent energy components) in the two directions $\overline{u_3^2}$ and $\overline{u_2^2}$. References 5 and 7 show that $\overline{u_2^2}$ proceeds rapidly but asymptotically toward zero at large velocity gradients, whereas $\overline{u_3^2}$ decreases slowly from the average of the energy components $\overline{u_i u_i}/3$, which increases with velocity gradient. Likewise from physical reasoning, it is clear that the thermal eddy diffusivity is greater in the direction of greater velocity fluctuations.

The existence of greater $\overline{u_3^2}$ than $\overline{u_2^2}$ has long been suspected [11] and, in recent times, has

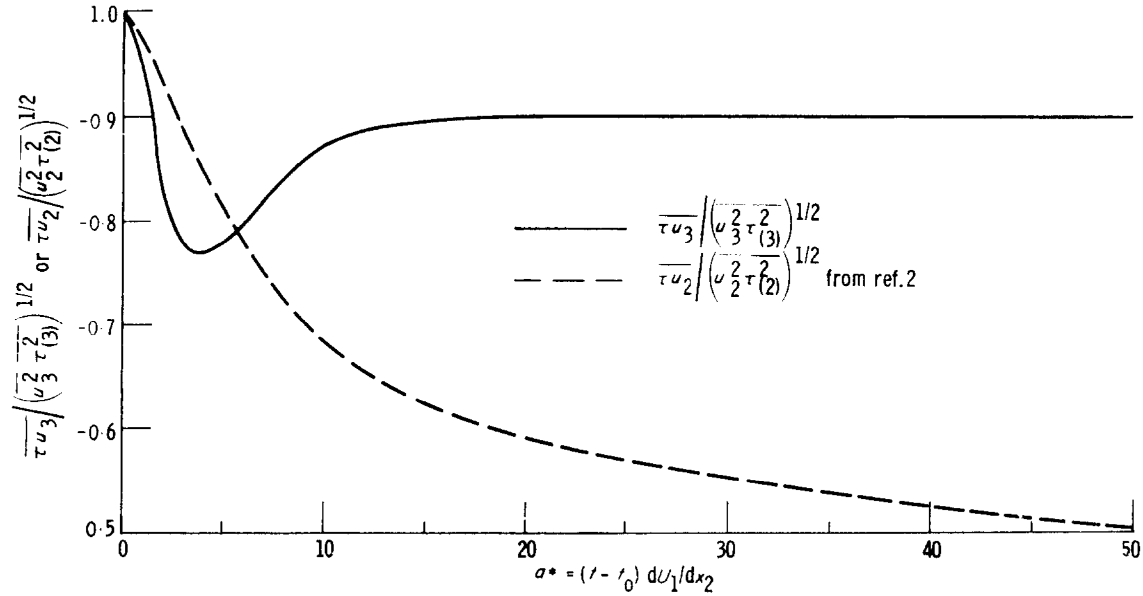


FIG. 5. Temperature–velocity correlation coefficients as a function of dimensionless velocity gradient. Prandtl number, 1.

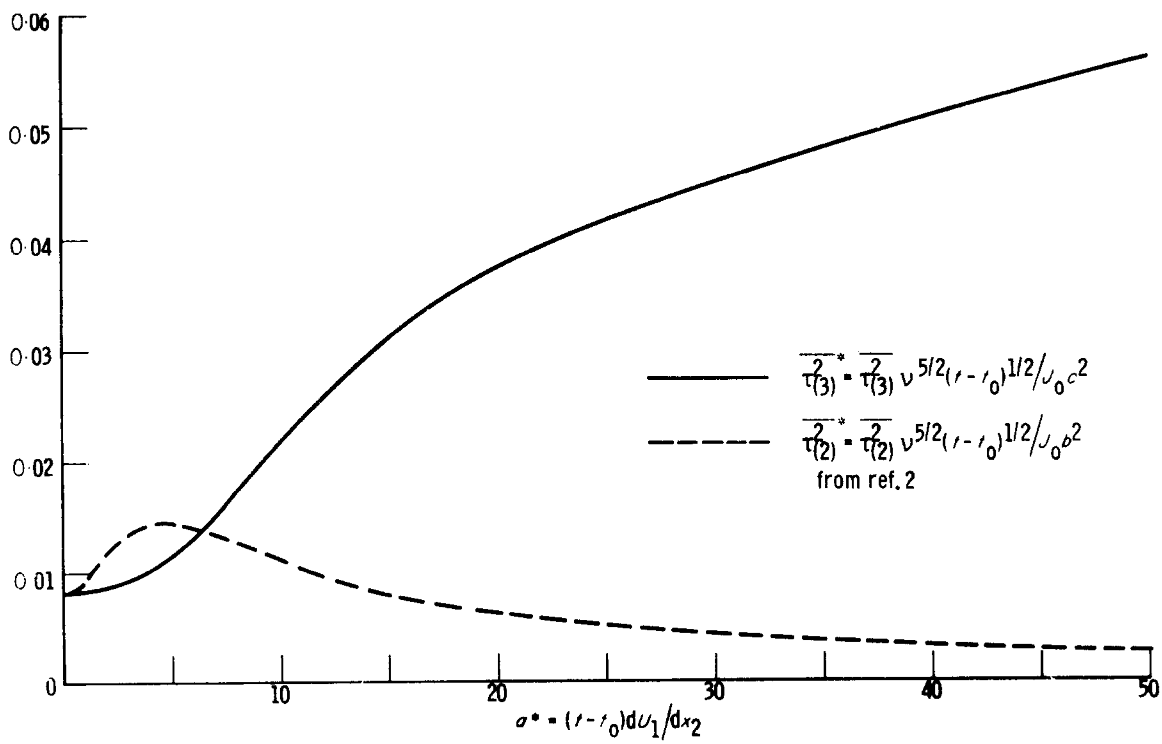


FIG. 6. Dimensionless $\overline{\tau^2}_{(2)}$ and $\overline{\tau^2}_{(3)}$ as a function of dimensionless velocity gradient. Prandtl number, 1.

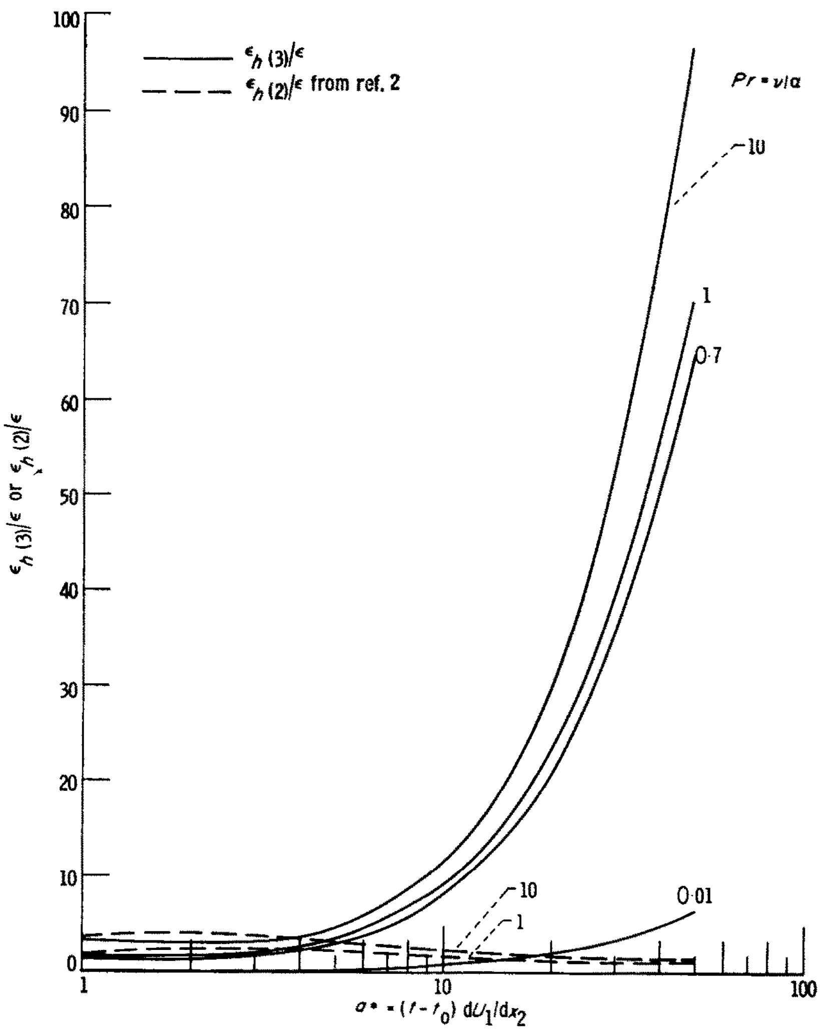


FIG. 7. Ratio of eddy diffusivity for heat transfer to that for momentum transfer as a function of dimensionless velocity gradient.

been verified experimentally in tube and channel flow [12], [13] and in boundary layers [14]. In fact, the ordering of all three components of turbulent energy (from largest to smallest) is the same ($\overline{u_1^2}$, $\overline{u_3^2}$, $\overline{u_2^2}$) in those measurements and in the present theory [7] at large velocity gradients. Apparently, not all features of boundary layers and tube flow are dependent on the presence of boundaries, which are absent in the theory.

Deissler [2] compared the transient analysis with a steady flow in a boundary layer or tube by taking $\kappa_{average}^* \sim 1$ from turbulent energy spectral curves and $0{\cdot}3\delta$ as a representative length, where δ is the boundary-layer thickness or the tube radius. If U is an average velocity and $dU_1/dx_2 \sim U/\delta$, then a^* is of the order of $0{\cdot}1\ U\delta/\nu$. This implies that $\epsilon_{h(3)}/\epsilon$ is larger than $\epsilon_{h(2)}/\epsilon$ for Reynolds numbers of 10^4 and over that are encountered in practice.

The results of the present analysis, together with existing velocity-fluctuation measurements, provide no support for an assumption of equal thermal eddy diffusivities in the radial and circumferential directions ($\epsilon_{h(2)} = \epsilon_{h(3)}$) in turbulent tube flow. Instead, the relation $\epsilon_{h(3)} > \epsilon_{h(2)}$ is indicated.

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Résumé—Des équations de corrélation pour des fluctuations statistiquement homogènes de vitesse et de température en deux points dans un écoulement infini de cisaillement uniforme sont obtenues en tenant compte d'un gradient de température dans une direction arbitraire dans un plan normal à la direction de l'écoulement. La turbulence isotrope excitée initialement décroît et devient anisotrope avec le temps. Après avoir introduit des transformations de Fourier, les équations spectrales résultantes sont résolues dans le cas d'une turbulence faible dans lequel les corrélations triples sont négligées par rapport aux corrélations doubles. Les spectres du transport de chaleur turbulent et de la fluctuation de température sont calculés. Pour de grands gradients de vitesse sans dimensions, la diffusivité thermique turbulente dans la direction normale au gradient de vitesse est beaucoup plus grande que celle dans la direction du gradient de vitesse. La diffusivité thermique turbulente dans la direction du gradient de vitesse tend à devenir égale à la diffusivité de quantité de mouvement turbulente à des gradients élevés de vitesse.

Zusammenfassung—Unter Berücksichtigung eines Temperaturgradienten von beliebiger Richtung in einer Ebene senkrecht zur Strömungsrichtung werden an zwei Stellen in einer unendlichen, gleichförmigen Scherströmung Korrelationsgleichungen für statisch homogene Geschwindigkeits- und Temperaturschwankungen abgeleitet. Die ursprünglich angeregte isotrope Turbulenz klingt ab und wird mit der Zeit anisotrop. Nach Einführung von Fourier-Transformationen werden die sich ergebenden Spektralgleichungen für den Fall schwacher Turbulenz gelöst, worin Dreifachkorrelationen im Vergleich zu Zweifachkorrelationen vernachlässigt werden. Spektra des Wärmeübergangs und der Temperaturschwankungen bei Turbulenz werden berechnet. Für grosse dimensionslose Geschwindigkeitsgradienten wird der turbulente Wärmeaustausch normal zum Geschwindigkeitsgradienten viel grösser als in Richtung des Geschwindigkeitsgradienten. Der turbulente Wärmeaustausch in Richtung des Geschwindigkeitsgradienten gleicht sich dem Impulsaustausch bei Turbulenz und bei grossen Geschwindigkeitsgradienten an.

Аннотация—В статье выводятся корреляционные уравнения для статистически гомогенных пульсаций скорости и температуры в двух точках неограниченного равномерного потока со сдвиговыми напряжениями в допущении температурного градиента в произвольном направлении в плоскости, перпендикулярной направлению течения. Вызванная вначале изотропная турбулентность вырождается и со временем становится анизотропной. После введения преобразований Фурье полученные спектральные уравнения решены для случая слабой турбулентности, где тройные корреляции пренебрегаются по сравнению с двойными корреляциями. Рассчитаны спектры турбулентных пульсаций теплообмена и температуры. При больших безразмерных градиентах скорости величина турбулентной температуропроводности в направлении, перпендикулярном градиенту скорости, намного больше, чем в направлении градиента скорости. Турбулентная температуропроводность в направлении градиента скорости стремится сравняться с коэффициентом турбулентной вязкости при больших градиентах скорости.